Nuclear Many-Body Problem with Nonuniform Density. I. Theory*

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The many-body problem for nuclear matter is considered under the constraint that the density of the system vary periodically in space; this is equivalent to the problem of an external periodic potential. The method of Martin, Schwinger, and Puff is employed to derive, through second order in the amplitude of the density ripple, a set of tractable equations for the energy of the system. No new approximations are needed beyond those already assumed for the uniform density case. The model exhibits some interesting complications of the finite nuclear problem, and yet possesses sufficient simplicity for solubility. In the long-wavelength limit, an expression obtains for the explicit dependence of the energy on the density gradient; this relationship is useful in a study of the nuclear surface problem. Numerical calculations are planned.

I. INTRODUCTION

NUCLEAR many-body theories have achieved considerable success in describing such properties of infinite systems as binding energy, density, compressibility, and symmetry energy, beginning with basic two-body forces.[1–3] This has stimulated various proposals to extend the methods to finite nuclei.[4] The complexities of the finite system are prodigious compared with the infinite case, and drastic simplifying assumptions have been employed to reduce the problem to tractable form. The most realistic of these efforts is the work of Brueckner, Lockett, and Rotenberg,[5] who performed Hartree-Fock calculations on the doubly closed shell nuclei O^{16}, Ca^{40}, and Zr^{90}. The "effective" two-body interaction for these calculations was the K matrix for infinite nuclear matter. The K matrix was assumed to be the same function of local density in the finite case as it is for the infinite, uniform density case.

The Brueckner, Lockett, and Rotenberg calculations achieved only moderate agreement with experiment. Although the same potentials reproduce well the properties of infinite nuclear matter, they yield, in the finite case, binding energies which are considerably too low and densities which are somewhat too high. The magnitude of the discrepancy in the binding energy is roughly half of the value of the surface energy in the semiempirical mass formula.

In order to obtain an understanding of the nuclear surface in general, and the finite nucleus in particular, it is at least necessary to investigate the effect of density gradients on the effective two-body interaction—and hence on the structure of the two-body correlations—as has already been noted by Brueckner, Lockett, and Rotenberg. A tool for their investigation is provided by the ripple model, which is discussed in the next section.

II. THE RIPPLE MODEL

We will consider an infinite nuclear system described by a many-body Hamiltonian H. The expectation value of the Hamiltonian is minimized subject to the constraint that the medium possess a static, periodic density variation, i.e., that the expectation value of the operator

$$Q=\cos qx \tag{2.1}$$

be a constant. This leads at once to the equivalent problem characterized by the Hamiltonian

$$\mathcal{H}=H+a_0Q, \tag{2.2}$$

where a_0 is a Lagrange multiplier, and a_0Q plays the role of an external, cosinusoidal potential. (In Sec. III we will also consider variations with the constraint that the number of particles be a constant.)

This problem exhibits many of the complications of finite nuclei and, at the same time possesses simplicities which render the solution tractable. In particular, the energy of the system can be calculated to second order in the constraining potential with no new approximations beyond those already assumed for the corresponding uniform density case. Calculations for nuclear deformation in finite nuclei have been formulated in a completely analogous manner. In that case, equilibrium deformations are found for a large class of nuclei; that is, in the absence of a constraint, the Hartree-Fock equations have a nonspherical solution. A corresponding phenomenon for nuclear matter would be the appearance of Overhauser[6] density correlations at a value of q near $2k_F$. On the basis of other investigations,[7] such do not appear to be likely and we have implicitly

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ignored the possibility in the present work. We have particular interest in the small q limit, but it would be interesting to see if the expansion in powers of a_0 does diverge for some value of q, corresponding to the appearance of Overhauser correlations.

In addition to the intrinsic interest in studying the properties of a soluble model which exhibits features of a finite system, the ripple model also provides information needed in the solution of the nuclear surface problem—which provided the original motivation. An outline of the relevance of the ripple model to the surface problem is as follows:

Let the ground-state energy of the nucleus be expanded in terms of gradients of the density:

$$E=\int d\tau\,\{\mathcal{E}(\rho)+f(\rho)|\nabla\rho|^2 +O[|\nabla\rho|^4,|\nabla\rho|^2\nabla^2\rho,(\nabla^2\rho)^2]\}, \quad (2.3)$$

where ρ is the particle density. No odd terms in the density gradient can occur, and a term of the form $g(\rho)\nabla^2\rho$ can be converted to $-g'(\rho)|\nabla\rho|^2$ by partial integration, and has already been included in Eq. (2.3). The terms of higher order in the gradient will not be considered.

In the presence of the external potential, $a_0\cos qx$, an otherwise uniform, infinite medium becomes polarized with a density given by

$$\rho=\bar{\rho}[1+c\cos qx+O(a_0^2)], \quad (2.5)$$

where c is proportional to a_0. To order c^2, the energy of the system, calculated from Eq. (2.3), is

$$E/N=\mathcal{E}(\bar{\rho})/\bar{\rho}+\tfrac{1}{4}c^2\bar{\rho}\mathcal{E}''(\bar{\rho})+\tfrac{1}{2}c^2q^2\bar{\rho}f(\bar{\rho})+\cdots. \quad (2.6)$$

Presumably the exact energy of the system can also be expanded to give

$$E/N=\langle H\rangle/N=\mathcal{E}(\bar{\rho})/\bar{\rho}+c^2A(\bar{\rho}) +c^2q^2B(\bar{\rho})+O(c^2q^4,c^4). \quad (2.7)$$

It is then a matter of identifying coefficients: $\mathcal{E}(\bar{\rho})/\bar{\rho}$ is the mean energy per particle in a uniform medium, as a function of the density. $A(\bar{\rho})=\frac{1}{4}\bar{\rho}\mathcal{E}''(\bar{\rho})$ is related to compressibility which, at normal density (subscript zero), is given by

$$K\equiv r_0^2\partial^2(E/N)/\partial r_0^2=36\rho_0A(\rho_0), \quad (2.8)$$

where $4\pi r_0^3/3=\rho_0^{-1}$. Finally, $f(\bar{\rho})=2\bar{\rho}^{-1}B(\bar{\rho})$.

Our program, therefore, is to calculate the density to first order and the energy to second order in the strength of the periodic potential, and both in the long-wavelength $[O(q^2)]$ limit. The many-body theory employed in the present work is a generalization of the method of Martin, Schwinger, and Puff.[8,2] The method is simple for calculational purposes, since it employs a reactance matrix which satisfies an integral equation without a projection operator. It appears, furthermore, to have been as successful as the Brueckner method in reproducing the properties of infinite nuclear matter.

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III. GENERALIZATION OF THE MARTIN-SCHWINGER-PUFF EQUATIONS

Consider a system of identical fermions described by the field operator $\psi(kt)$, where k denotes the three components of momentum, and the spin and isospin indices. The Fermi statistics are fixed by the equal-time anticommutation relations,

$$[\psi(kt),\psi^\dagger(k't)]_+=\delta_{kk'}. \quad (3.1)$$

The interaction is described by the Hamiltonian operator (in units $\hbar=2m=1$)

$$H=\int_k\psi^\dagger(kt)k^2\psi(kt)+\tfrac{1}{4}\int_{k_1k_2k_3k_4}\psi^\dagger(k_1t) \times\psi^\dagger(k_2t)(k_1k_2|v|k_3k_4)\psi(k_4t)\psi(k_3t), \quad (3.2)$$

where $\int_k$ denotes integration over all momentum space $\int d^3k/(2\pi)^3$, as well as summation over spin and isospin indices, and $\delta_{kk'}=(2\pi)^3\delta(k-k')\delta_{\sigma\sigma'}\delta_{\tau\tau'}$. Thus $\int_k\delta_{kk'}=\int d^3k\,\delta(k-k')=1$. That is, $\int_k$ behaves like the sum $\sum_k$, and $\delta_{kk'}$ behaves like the ordinary Kronecker delta. The matrix elements of the potential v are antisymmetric both in the first and in the second pair of arguments, and are equal to the usual direct integral minus the exchange integral.

We wish to minimize the expectation value of the Hamiltonian, subject to the constraint that the expectation value of the number operator

$$N=\int_k\psi^\dagger(kt)\psi(kt), \quad (3.3)$$

and of the operator

$$Q=\int_{kk'}\psi^\dagger(kt)(k|u|k')\psi(k't) \quad (3.4)$$

be constant. (We shall use, later, the specific form $u=\cos qx$.) This is equivalent to minimizing the expectation value of

$$\mathcal{H}=H+a_0Q-\mu N, \quad (3.5)$$

where a_0 and μ are Lagrange multipliers. The term a_0Q acts as an external potential which induces density variations, and μ plays the usual role of a chemical potential.

The field operators obey the equation of motion

$$i\frac{\partial}{\partial t}\psi(kt)=[\psi(kt),\mathcal{H}]. \quad (3.6)$$

We follow Martin, Schwinger, and Puff[8,2] in defining the *n*-particle Green's *operator*

$$\mathcal{G}_n(1t_1\cdots nt_{n1};\,1't_1'\cdots n't_n')\equiv(-i)^n\epsilon(t_1\cdots t_nt_n'\cdots t_1') \times[\psi(1t_1)\cdots\psi(nt_n)\psi^\dagger(n't_n')\cdots\psi^\dagger(1't_1')]_+, \quad (3.7)$$

where + means time ordering of the field operators, earliest time to the right, and $\epsilon(t_1\cdots t_1')$ is the sign of the permutation required to perform the time ordering from the given ordering. We have introduced the abbreviation j for k_j.

An infinite set of coupled differential equations for the $\mathcal{G}_n$ is obtained from (3.7) by use of Eq. (3.6), and the anticommutation relation (3.1):

$$\int_{1''}\left[\left(i\frac{\partial}{\partial t_1}-k_1^2+\mu\right)\delta_{11''}-a_0(1|u|1'')\right]\times\mathcal{G}_n(1''t_1 2t_2\cdots n't_n')$$
$$=\sum_{j'=1}^{n}(-)^{j'+1}\delta_{1j'}\delta(t_1-t_{j'})$$
$$\times\mathcal{G}_{n-1}(2t_2\cdots nt_n;\,1't_1'\cdots(\text{omit } j't_j')\cdots n't_n')$$
$$+i(-)^n\tfrac{1}{2}\int_{2''3''4''}(12''|v|3''4'')$$
$$\times\mathcal{G}_{n+1}(3''t_1,4''t_1,2t_2,\cdots nt_n;\,1't_1'\cdots n't_n'2''t_1^+),\quad(3.8)$$

where $\mathcal{G}_0\equiv 1$, and t_1^+ is a time infinitesimally larger than t_1. This set may be transformed to integral form,

$$\mathcal{G}_n(1t_1\cdots n't_n')=\sum_{j'=1}^{n}(-)^{j'+1}\mathcal{G}_1^0(1t_1;j't_j')$$
$$\times\mathcal{G}_{n-1}(2t_2\cdots nt_n;\,1't_1'\cdots(\text{omit } j't_j')\cdots n't_n')$$
$$+i(-)^n\tfrac{1}{2}\int dt\int_{1''2''3''4''}\mathcal{G}_1^0(1t_1;\,1''t)(1''2''|v|3''4'')$$
$$\times\mathcal{G}_{n+1}(3''t,4''t,2t_2\cdots nt_n;\,1't_1'\cdots n't_n'2''t^+),\quad(3.9)$$

where we have introduced an interaction-free Green's operator, $\mathcal{G}_1^0$, which satisfies

$$\left(i\frac{\partial}{\partial t_1}-k_1^2+\mu\right)\mathcal{G}_1^0(1t_1;\,1't_1')$$
$$-a_0\int_{1''}(1|u|1'')\mathcal{G}_1^0(1''t_1;\,1't_1')=\delta_{11'}\delta(t_1-t_1').\quad(3.10)$$

We now define the average value of an operator X by

$$\langle X\rangle\equiv\frac{\text{Trace}\exp(-i\tau\mathcal{H})X}{\text{Trace}\exp(-i\tau\mathcal{H})},\quad(3.11)$$

where τ is an artificial time parameter. For imaginary values of τ, $i\tau$ plays the role of the inverse temperature $\beta=(kT)^{-1}$. We shall eventually take the limit $i\tau\to\infty$ (zero temperature), in which case the trace average (3.11) becomes an average over the states of lowest eigenvalue of $\mathcal{H}$. This is equivalent to an average over the states of lowest eigenvalue of H which have a given average value of N and of Q, as discussed above.

We define the n-body Green's *function* as the average value of the n-body Green's operator, through (3.11):

$$G_n(1t_1\cdots n't_n')\equiv\langle\mathcal{G}_n(1t_1\cdots n't_n')\rangle.\quad(3.12)$$

The average values of N, Q, and H are given by

$$\langle N\rangle=-i\int_1 G_1(1t;\,1t^+),\quad(3.13)$$

$$\langle Q\rangle=-i\int_{11'}(1|u|1')G_1(1't;\,1t^+),\quad(3.14)$$

$$\langle H\rangle=-i\int_1 k^2G_1(1t;\,1t^+)$$
$$-\tfrac{1}{4}\int_{1234}(12|v|34)G_2(1t2t;\,3t^+4t^+).\quad(3.15)$$

By use of (3.8) with $n=1$, this last becomes simply

$$\langle H\rangle=-i\int_{11'}\tfrac{1}{2}\left[\left(i\frac{\partial}{\partial t}+k_1^2+\mu\right)\delta_{11'}-a_0(1|u|1')\right]G_1(1't;\,1t^+).\quad(3.16)$$

The one-particle Green's function may be written

$$G_1(1t;\,1't')=G_+(1t:\,1't')\theta(t-t')+G_-(1t;\,1't')\theta(t'-t),\quad(3.17)$$

where

$$\theta(y)=1,\quad y>0,$$
$$=0,\quad y<0,$$

and G_+ and G_- are defined as the averages (3.11) of the operators

$$\mathcal{G}_+(1t;\,1't')=-i\psi(1t)\psi^\dagger(1't'),$$
$$\mathcal{G}_-(1t;\,1't')=i\psi^\dagger(1't')\psi(1t),\quad(3.18)$$

each of which is defined for all t, t'.

Since $\mathcal{H}$ is time independent, the time coordinates of the one-body Green's functions appear only in the combination $t-t'$. Thus we may consider the Fourier transforms

$$G_\pm(1t;\,1't')=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}G_\pm(11',\omega).\quad(3.19)$$

From the invariance of the trace (3.11) under cyclic permutation of the operators, and from the time translation of the field operators

$$\psi(1,\,t+\tau)=\exp(i\mathcal{H}\tau)\psi(1t)\exp(-i\mathcal{H}\tau),\quad(3.20)$$

it can be shown[8] that G_+ and G_- are related by

$$G_+(11',\omega)(1+e^{-i\omega\tau})=-G_-(11',\omega)(1+e^{i\omega\tau})$$
$$\equiv-iA(11',\omega).\quad(3.21)$$

This defines the spectral function, $A(11'\omega)$, so that the Fourier transform of G_1 can be written as

$$G_1(11',\omega)=\lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty}\frac{d\sigma}{2\pi}A(11',\sigma)\Big[\frac{1}{(1+e^{-i\sigma\tau})(\omega-\sigma+i\epsilon)}+\frac{1}{(1+e^{i\sigma\tau})(\omega-\sigma-i\epsilon)}\Big]. \quad (3.22)$$

At this point we take the zero temperature limit $i\tau\to\infty$. Then

$$(1+e^{\pm i\sigma\tau})^{-1}\to\theta(\mp\sigma),$$
$$G_\pm(11',\omega)=\mp iA(11',\omega)\theta(\pm\omega), \quad (3.23)$$

and

$$G_1(11',\omega)=\lim_{\epsilon\to 0^+}\Big\{\int_0^{\infty}\frac{d\sigma}{2\pi}\frac{A(11',\sigma)}{\omega-\sigma+i\epsilon}+\int_{-\infty}^{0}\frac{d\sigma}{2\pi}\frac{A(11',\sigma)}{\omega-\sigma-i\epsilon}\Big\}. \quad (3.24)$$

Thus the artificial time τ, introduced in (3.11), is eliminated; from here on we consider the zero-temperature limit.

We may now write

$$G_1(1t;1't^+)=G_-(1t;1't)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}G_-(11',\omega)=i\int_{-\infty}^{0}\frac{d\omega}{2\pi}A(11',\omega). \quad (3.25)$$

It follows that the averages (3.13), (3.14), and (3.16) can be expressed in terms of the spectral function:

$$\langle N\rangle=\int_1\int_{-\infty}^{0}\frac{d\omega}{2\pi}A(11,\omega), \quad (3.26)$$

$$\langle Q\rangle=\int_{11'}\int_{-\infty}^{0}\frac{d\omega}{2\pi}(1|u|1')A(1'1,\omega), \quad (3.27)$$

$$\langle H\rangle=\tfrac{1}{2}\int_{11'}\int_{-\infty}^{0}\frac{d\omega}{2\pi}[(\omega+k_1^2+\mu)\delta_{11'}-a_0(1|u|1')]A(1'1,\omega). \quad (3.28)$$

So far we have considered only real values of ω. If, however, we define the analytic continuation of $G_1(11',\omega)$ into the complex ω plane as

$$\hat{G}_1(11',\omega)=\int_{-\infty}^{\infty}\frac{d\sigma}{2\pi}\frac{A(11',\sigma)}{\omega-\sigma}, \quad (3.29)$$

then it follows that (3.24) can be inverted to give

$$A(11',\omega)=i\lim_{\eta\to 0^+}[\hat{G}_1(11',\omega+i\eta)-\hat{G}(11',\omega-i\eta)]. \quad (3.30)$$

The coupled set of Green's operator equations (3.8) may be rewritten for the Green's functions, starting with

$$\int_{1''}\Big[\Big(i\frac{\partial}{\partial t_1}-k_1^2+\mu\Big)\delta_{11''}-a_0(1|u|1'')\Big]\times G_1(1''t_1;1't_1')=\delta_{11'}\delta(t_1-t_1')-i\tfrac{1}{2}\int_{234}(12|v|34)G_2(3t_14t_1;1't_1'2t_1^+), \quad (3.31)$$

and from (3.10) we get

$$\int_{1''}\Big[\Big(i\frac{\partial}{\partial t_1}-k_1^2+\mu\Big)\delta_{11''}-a_0(1|u|1'')\Big]\times G_1^0(1''t_1;1't_1')=\delta_{11'}\delta(t_1-t_1'). \quad (3.32)$$

The integral equations (3.9) can be rewritten

$$G_1(1t_1;1't_1')=G_1^0(1t_1;1't_1')-i\tfrac{1}{2}\int dt\int_{1''234}G_1^0(1t_1;1''t)(1''2|v|34)\times G_2(3t4t;1't_1'2't^+), \quad (3.33)$$

$$G_2(1t_12t_2;1't_1'2't_2')=G_1^0(1t_1;1't_1')G_1(2t_2;2't_2')-G_1^0(1t_1;2't_2')G_1(2t_2;1t_2')+i\tfrac{1}{2}\int dt\int_{1''2''3''4''}G_1^0(1t_1;1''t)(1''2''|v|3''4'')\times G_3(3''t4''t2t_2;1't_1'2't_2'2''t^+), \quad (3.34)$$

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The series is terminated by making the Martin-Schwinger-Puff approximations: First G_3 in the integral (3.34) is approximated by

$$G_3(3''4''2;1'2'2'')=G_2(3''4'';1'2')G_1(2;2'')-G_2(3''4'';1'2'')G_1(2;2')+G_2(3''4'';2'2'')G_1(2;1'), \quad (3.35)$$

where the times are understood. The second approximation is to replace $G_1(2;2'')$ by $G_1^0(2;2'')$ in (3.35). Combining (3.33), (3.34), and (3.35) leads to an equation for G_2,

$$G_2(1t_12t_2;1't_1'2't_2')=G_1(1t_1;1't_1')G_1(2t_2;2't_2')-G_1(1t_1;2't_2')G_1(2t_2;1't_1')+i\tfrac{1}{2}\int dt\int_{1''2''3''4''}G_1^0(1t_1;1''t)G_1^0(2t_2;2''t)\times(1''2''|v|3''4'')G_2(3''t4''t;1't_1'2't_2'), \quad (3.36)$$

which, together with (3.33), form a closed pair of integral equations. The approximation (3.35) means that only the two particles interacting through the potential in G_3, (3 in. and 4 in.), are explicitly correlated

(except for antisymmetry). The second approximation is less physical, but simplifies the calculation in later stages because the Fourier transform of G_1^0 is non-singular for negative ω. The two approximations together can be shown to be equivalent to the somewhat unsymmetric approximation

$$\int_{1''2''3''4''}(12|v|1''2'')(34|v|3''4'')\times G_4(1''2''3''4'';1'2'3'4')\approx\int_{1''2''3''4''}(12|v|1''2'')\times(34|v|3''4'')[G_2(1''2'';2'3')G_2(3''4'';1'4')-G_2(1''2'';2'4')G_2(3''4'';1'3')]. \quad (3.37)$$

For the solution of Eqs. (3.33) and (3.36) we require only G_2 with the first two times equal. By *ansatz*, we introduce a matrix Ω such that

$$G_2(1t_1 2t_1;1't_1'2't_2')=\int dt\int_{34}(12|\Omega(t_1-t)|34)\times G_1(3t;1't_1')G_1(4t;2't_2'). \quad (3.38)$$

Then (3.36) is satisfied if Ω satisfies

$$(12|\Omega(t_1-t_1')|1'2')=\delta(t_1-t_1')[\delta_{11'}\delta_{22'}-\delta_{12'}\delta_{21'}]+i\tfrac{1}{2}\int dt\int_{1''2''3''4''}G_1^0(1t_1,1''t)G_1^0(2t_1,2''t)\times(1''2''|v|3''4'')(3''4''|\Omega(t-t_1')|12). \quad (3.39)$$

If we now define

$$(12|T(t_1-t_1')|1'2')=\tfrac{1}{2}\int_{34}(12|v|34)(34|\Omega(t_1-t_1')|1'2'), \quad (3.40)$$

then from (3.39) we obtain

$$(12|T(t_1-t_1')|1'2')=\delta(t_1-t_1')(12|v|1'2')+i\tfrac{1}{2}\int dt\int_{1''2''3''4''}(12|v|1''2'')G_1^0(1''t_1;3''t)\times G_1^0(2''t_1;4''t)(3''4''|T(t-t_1')|1'2'). \quad (3.41)$$

Taking the Fourier transform in t_1-t_1', $T(t_1-t_1')\equiv\int T(\omega+2\mu)e^{-i\omega(t_1-t_1')}(d\omega/2\pi)$, we find

$$(12|T(\omega+2\mu)|1'2')=(12|v|1'2')+\tfrac{1}{2}\int_{1''2''3''4''}(12|v|1''2'')\Lambda(1''2''3''4'',\omega)\times(3''4''|T(\omega+2\mu)|1'2'), \quad (3.42)$$

where

$$\Lambda(1''2''3''4'',\omega)=i\int_{-\infty}^{\infty}d(t_1-t)e^{i\omega(t_1-t)}\times G_1^0(1''t_1;3''t)G_1^0(2''t_1;4''t). \quad (3.43)$$

The Fourier transform of (3.31) becomes

$$\int_{1''}[(\omega_1-k_1^2)\delta_{11''}-a_0(1|u|1'')]G_1(1''1',\omega_1)=\delta_{11'}-i\lim_{\theta\to0^+}\int\frac{d\omega_2}{2\pi}\int_{21''2''}(12|T(\hat\omega_1+\hat\omega_2)|1''2'')\times G_1(1''1',\omega_1)G_1(2''2,\omega_2)e^{i\omega_2\theta}, \quad (3.44)$$

where we have introduced $\hat\omega_1\equiv\omega_1+\mu$. We define the self-consistent one-body potential by

$$(1|U(\hat\omega_1)|1'')\equiv-i\lim_{\theta\to0^+}\int\frac{d\omega_2}{2\pi}\times\int_{2''2}(12|T(\hat\omega_1+\hat\omega_2)|1''2'')G_1(2''2,\omega_2)e^{i\omega_2\theta}. \quad (3.45)$$

We perform the ω_2 integration, assuming that $T(\hat\omega_1+\hat\omega_2)$ has poles only in the lower ω_2 half-plane: This can be shown to be true for a large class of potentials. Then, using (3.24), we obtain

$$(1|U(\hat\omega_1)|1'')=\int_{-\infty}^{0}\frac{d\omega_2}{2\pi}\int_{22''}(12|T(\hat\omega_1+\hat\omega_2)|1''2'')\times A(2''2,\omega_2). \quad (3.46)$$

We may rewrite (3.44) in the form

$$\int_{1''}[(\hat\omega_1-k_1^2)\delta_{11''}-a_0(1|u|1'')-(1|U(\hat\omega_1)|1'')]G_1(1''1',\omega_1)=\delta_{11'}. \quad (3.47)$$

We now define an effective one-body Hamiltonian,

$$(1|h(\hat\omega)|1')=k_1^2\delta_{11'}+a_0(1|u|1')+(1|U(\hat\omega)|1'). \quad (3.48)$$

Now, if $h(\hat\omega)$ is Hermitian (as will be shown for $\hat\omega$ of interest), an eigenvalue equation for a given value of $\hat\omega$ may be written as

$$\int_{1'}(1|h(\hat\omega)1')\psi_m(1'\hat\omega)=\mathcal{E}_m(\hat\omega)\psi_m(1,\hat\omega). \quad (3.49)$$

By taking matrix elements on both sides, Eq. (3.47) yields, in the representation $\psi_m(1,\hat\omega)=(1|m,\hat\omega\rangle$,

$$[\hat\omega-\mathcal{E}_m(\hat\omega)]\langle m,\hat\omega|G_1(\omega)|n,\hat\omega\rangle=\delta_{mn}. \quad (3.50)$$

Clearly, except at the singularities $\hat\omega=\mathcal{E}_m(\hat\omega)$,

$$G_1(11',\omega)=\sum_m\frac{(1|m,\hat\omega\rangle\langle m,\hat\omega|1')}{\hat\omega-\mathcal{E}_m(\hat\omega)}. \quad (3.51)$$

The analytic continuation of G_1, $\hat{G}_1(11',\omega)$, also satisfies (3.51), where $(1|m,\hat{\omega}\rangle$ and $\mathcal{E}_m(\hat{\omega})$ are analytically continued into the complex $\hat{\omega}$ plane. Then (3.30) can be written

$$A(11',\omega)=i\lim_{\eta\to 0^+}\sum_m\left\{\frac{(1|m,\hat{\omega}+i\eta\rangle\langle m,\hat{\omega}+i\eta|1')}{\hat{\omega}-\mathcal{E}_m(\hat{\omega}+i\eta)+i\eta}\right.$$
$$\left.-\frac{(1|m,\hat{\omega}-i\eta\rangle\langle m,\hat{\omega}-i\eta|1')}{\hat{\omega}-\mathcal{E}_m(\hat{\omega}-i\eta)-i\eta}\right\}$$
$$=2\pi\textstyle\sum_m\rho_m(\hat{\omega})(1|m\rangle\langle m|1')\delta(\hat{\omega}-E_m), \quad (3.52)$$

where E_m is the value of $\hat{\omega}$ for which

$$\hat{\omega}=\mathcal{E}_m(\hat{\omega}), \quad (3.53)$$

and

$$\rho_m(\hat{\omega})=[1-(d/d\hat{\omega})\mathcal{E}_m(\hat{\omega})]^{-1}. \quad (3.54)$$

We have defined $(1|m\rangle\equiv(1|m,E_m\rangle$, and shall also use $\rho_m\equiv\rho_m(E_m)$.

$G_1^0(11',\omega)$ is the solution of (3.47) with no interaction term, $(1|\mathcal{U}(\hat{\omega})|1')$. Thus, if

$$\int_{1''}[k_1^2\delta_{11''}+a_0(1|u|1')]\phi_\nu(1'')=\eta_\nu\phi_\nu(1), \quad (3.55)$$

then, from (3.52) and (3.24), we obtain

$$G_1^0(11',\omega_1)=\lim_{\epsilon\to 0^+}\sum_\nu\phi_\nu(1)\phi_\nu^*(1')$$
$$\times\left(\frac{\theta(\eta_\nu-\mu)}{\omega_1+\mu-\eta_\nu+i\epsilon}+\frac{\theta(\mu-\eta_\nu)}{\omega_1+\mu-\eta_\nu-i\epsilon}\right) \quad (3.56)$$

which, for $\mu<0$ (bound system), and a_0 small enough that $\eta_\nu>\mu$, becomes

$$G_1^0(11',\omega_1)=\lim_{\epsilon\to 0^+}\sum_\nu\frac{\phi_\nu(1)\phi_\nu^*(1')}{\omega_1+\mu-\eta_\nu+i\epsilon}. \quad (3.57)$$

Λ is obtained from (3.43) by convolution

$$\Lambda(1234,\omega)=i\int_{-\infty}^{\infty}\frac{d\sigma}{2\pi}G_1^0(13,\sigma)G_1^0(24,\omega-\sigma)$$
$$=\lim_{\epsilon\to 0^+}\sum_{\nu,\lambda}\frac{\phi_\nu(1)\phi_\nu^*(3)\phi_\lambda(2)\phi_\lambda^*(4)}{\omega-\eta_\nu-\eta_\lambda+2\mu+i\epsilon}. \quad (3.58)$$

This can also be written as a matrix element of an operator

$$\Lambda(1234,\omega)=(12|\Lambda(\omega)|34),$$
$$\Lambda(\omega)=\lim_{\epsilon\to 0^+}\frac{1}{\omega-H_0-a_0u+2\mu+i\epsilon}, \quad (3.59)$$

where

$$(12|H_0|1'2')=(k_1^2+k_2^2)\delta_{11'}\delta_{22'}$$

and

$$(12|u|1'2')=(1|u|1')\delta_{22'}+(2|u|2')\delta_{11'}.$$

Then Eq. (3.42) can be written in operator form

$$T(\omega+2\mu)=v+\tfrac{1}{2}v\frac{1}{\omega-H_0-a_0u+2\mu+i\epsilon}T(\omega+2\mu). \quad (3.60)$$

It can be shown that $T(\omega)$, satisfying (3.60), has poles only approaching the real ω axis from below. (See, for example, Watson.[9]) Also, it is clear that for $\omega+2\mu$ less than the lowest eigenvalue of H_0+a_0u, T is real. For a_0 small enough that the lowest eigenvalue is still greater than 2μ, T will be real for negative ω. Thus $h(\hat{\omega})$ *is* Hermitian for negative ω, which is the region of interest.

Equations (3.26), (3.27), and (3.28) can be rewritten, using (3.52), as

$$\langle N\rangle=\textstyle\sum_m\int_1\rho_m(1|m\rangle\langle m|1)\theta(\mu-E_m)$$
$$=\textstyle\sum_m\rho_m\theta(\mu-E_m), \quad (3.61)$$

$$\langle Q\rangle=\textstyle\sum_m\int_{11'}\rho_m\langle m|1)(1|u|1')(1'|m\rangle\theta(\mu-E_m)$$
$$=\textstyle\sum_m\rho_m\langle m|u|m\rangle\theta(\mu-E_m), \quad (3.62)$$

$$\langle H\rangle=\tfrac{1}{2}\textstyle\sum_m\int_{11'}\rho_m\langle m|1)[(E_m+k_1^2)\delta_{11'}$$
$$-a_0(1|u|1')](1'|m\rangle\theta(\mu-E_m)$$
$$=\tfrac{1}{2}\textstyle\sum_m\rho_m\langle m|[E_m+k^2-a_0u]|m\rangle\theta(\mu-E_m). \quad (3.63)$$

These quantities can be calculated with knowledge of $(1|m\rangle$, ρ_m, and E_m, which are, in turn, obtained by the simultaneous solution of

$$\int_{1'}\{k_1^2\delta_{11'}+a_0(1|u|1')+(1|\mathcal{U}(\hat{\omega})|1')\}(1'|m,\hat{\omega}\rangle$$
$$=\mathcal{E}_m(\hat{\omega})(1|m,\hat{\omega}\rangle, \quad (3.64)$$

$$(1|\mathcal{U}(\hat{\omega})|1')$$
$$=\textstyle\sum_{m'}\int_{22'}\langle m'|2)(12|T(\hat{\omega}+E_{m'})|1'2')(2'|m'\rangle$$
$$\times\rho_{m'}\theta(\mu-E_{m'})=\textstyle\sum_{m'}(1|\langle m'|T(\hat{\omega}+E_{m'})|m'\rangle|1')$$
$$\times\rho_{m'}\theta(\mu-E_{m'}), \quad (3.65)$$

$$\rho_m=\left[1-\frac{d}{d\hat{\omega}}\mathcal{E}_m(\hat{\omega})\right]^{-1}\Bigg|_{\hat{\omega}=E_m}, \quad (3.66)$$

$$E_m=\mathcal{E}_m(E_m). \quad (3.67)$$

These equations are simply obtained from (3.49), (3.46), (3.54), and (3.53), respectively.

[9] K. M. Watson, Phys. Rev. **103**, 489 (1956).

IV. EQUATIONS THROUGH SECOND ORDER

We now use the specific form for the constraining potential, which in configuration space is $u(x)=\cos qx$, and in momentum space is

$$(1|u|1')=\tfrac{1}{2}[\delta_{1,1'+q}+\delta_{1,1'-q}]. \tag{4.1}$$

We calculate the energy (3.63) and density (3.61) to second order in a_0. For this purpose, we expand all quantities in powers of a_0:

$$\begin{aligned} T&=T^{(0)}+a_0T^{(1)}+a_0^2T^{(2)}+\cdots,\\ \psi_m&=\psi_m{}^{(0)}+a_0\psi_m{}^{(1)}+a_0^2\psi_m{}^{(2)}+\cdots,\\ E_m&=E_m{}^{(0)}+0+a_0^2E_m{}^{(2)}+\cdots, \quad \text{etc.} \end{aligned} \tag{4.2}$$

The first term in each expansion corresponds to the quantities derived by Puff for the uniform density case. In general, scalar quantities such as ρ_m, E_m, etc., do not contain odd powers of a_0.

Consider first the T matrix, which satisfies the equation, for $\omega<+2\mu$,

$$T(\omega)=v+\tfrac{1}{2}v\frac{1}{\omega-H_0-a_0u}T(\omega). \tag{3.60'}$$

We expand the propagator in powers of a_0 according to

$$\begin{aligned} (\omega-H_0-a_0u)^{-1}&=(\omega-H_0)^{-1}+a_0(\omega-H_0)^{-1}u(\omega-H_0)^{-1}\\ &\quad+a_0^2(\omega-H_0)^{-1}u(\omega-H_0)^{-1}u(\omega-H_0)^{-1}. \end{aligned} \tag{4.3}$$

Then

$$T^{(0)}(\omega)=v+\tfrac{1}{2}v\frac{1}{\omega-H_0}T^{(0)}(\omega), \tag{4.4a}$$

or

$$T^{(0)}(\omega)=(\omega-H_0)(\omega-H_0-\tfrac{1}{2}v)^{-1}v. \tag{4.4b}$$

Similarly

$$T^{(1)}(\omega)=\tfrac{1}{2}v(\omega-H_0)^{-1}[u(\omega-H_0)^{-1}T^{(0)}(\omega)+T^{(1)}(\omega)], \tag{4.5a}$$

or

$$\begin{aligned} T^{(1)}(\omega)&=(\omega-H_0)(\omega-H_0-\tfrac{1}{2}v)^{-1}\tfrac{1}{2}v(\omega-H_0)^{-1}\\ &\quad\times u(\omega-H_0)^{-1}T^{(0)}(\omega)\\ &=\tfrac{1}{2}T^{(0)}(\omega)(\omega-H_0)^{-1}u(\omega-H_0)^{-1}T^{(0)}(\omega), \end{aligned} \tag{4.5b}$$

and

$$T^{(2)}(\omega)=\tfrac{1}{2}v(\omega-H_0)^{-1}[u(\omega-H_0)^{-1}u(\omega-H_0)^{-1}T^{(0)}(\omega)+u(\omega-H_0)^{-1}T^{(1)}(\omega)+T^{(2)}(\omega)], \tag{4.6a}$$

or

$$\begin{aligned} T^{(2)}(\omega)&=\tfrac{1}{2}T^{(0)}(\omega)(\omega-H_0)^{-1}u[(\omega-H_0)^{-1}\\ &\quad+\tfrac{1}{2}(\omega-H_0)^{-1}T^{(0)}(\omega)(\omega-H_0)^{-1}]\\ &\quad\times u(\omega-H_0)^{-1}T^{(0)}(\omega). \end{aligned} \tag{4.6b}$$

Thus once the integral equation for $T^{(0)}$ has been solved, $T^{(1)}$ and $T^{(2)}$ can be obtained by explicit integration.

Next we turn to the solution of the wave function equation,

$$\int_{k'}\{[k^2-\mathcal{E}_m(\hat{\omega})]\delta_{kk'}+a_0(k|u|k')+(k|\mathcal{U}(\hat{\omega})|k')\}\psi_m(k',\hat{\omega})=0. \tag{3.64'}$$

Clearly,

$$\psi_m{}^{(0)}(k,\hat{\omega})=\delta_{mk}=\phi_m(k), \tag{4.7a}$$

where the label m denotes the momentum and discrete quantum numbers of the unperturbed function. We now make the *ansatz*

$$\psi_m{}^{(1)}(k,\hat{\omega})=\textstyle\sum_{\Delta=\pm q}(m-\Delta|\alpha(\hat{\omega})|m)\phi_{m-\Delta}(k), \tag{4.7b}$$

$$\psi_m{}^{(2)}(k,\hat{\omega})=\beta_m(\hat{\omega})\phi_m(k)+\textstyle\sum_{\Delta=\pm q}(m-2\Delta|\gamma(\hat{\omega})|m)\phi_{m-2\Delta}(k). \tag{4.7c}$$

We will not, in fact, require γ.

From Eq. (3.65), we can write down the expansion for the self-consistent one-body potential. This is

$$(k|\mathcal{U}^{(0)}(\hat{\omega})|k')=\int_m\hat{\rho}_m{}^{(0)}(km|T^{(0)}(\hat{\omega}+E_m{}^{(0)})|k'm), \tag{4.8a}$$

which has only diagonal elements. We have introduced the notation

$$\hat{\rho}_m{}^{(0)}=\theta(\mu-E_m{}^{(0)})\rho_m{}^{(0)},$$

and we have replaced $\sum_m$ by $\int_m$. $\mathcal{U}^{(1)}$ has only non-diagonal elements of the form

$$\begin{aligned} &(k|\mathcal{U}^{(1)}(\hat{\omega})|k+\Delta)\\ &=\int_m\hat{\rho}_m{}^{(0)}\{(km|T^{(1)}(\hat{\omega}+E_m{}^{(0)})|k+\Delta,m)\\ &\quad+(km|T^{(0)}(\hat{\omega}+E_m{}^{(0)})|k+\Delta,m-\Delta)\\ &\quad\times(m-\Delta|\alpha(E_m{}^{(0)})|m)\\ &\quad+(k,m+\Delta|T^{(0)}(\hat{\omega}+E_m{}^{(0)})|k+\Delta,m)\\ &\quad\times(m+\Delta|\alpha(E_m{}^{(0)})|m)^*\}, \end{aligned} \tag{4.8b}$$

and is Hermitian in the momentum variables. Only the diagonal second-order elements of $\mathcal{U}$ are required, and these do not affect the wave functions to second order; $\mathcal{U}^{(2)}$ is obtained later.

By application of standard perturbation theory, the solution to the wave-function equation (3.64′) can be written through second order as

$$\begin{aligned} \psi_m(k,\hat{\omega})&=\left\{1-\frac{a_0^2}{2}\sum_{\Delta=\pm q}\frac{(m-\Delta|W(\hat{\omega})|m)^2}{[\mathcal{E}_m{}^{(0)}(\hat{\omega})-\mathcal{E}_{m-\Delta}{}^{(0)}(\hat{\omega})]^2}\right\}\phi_m(k)\\ &\quad+a_0\sum_{\Delta=\pm q}\frac{(m-\Delta|W(\hat{\omega})|m)}{\mathcal{E}_m{}^{(0)}(\hat{\omega})-\mathcal{E}_{m-\Delta}{}^{(0)}(\hat{\omega})}\phi_{m-\Delta}(k)\\ &\quad+O(a_0^2)\phi_{m\pm2\Delta}(k), \end{aligned} \tag{4.9}$$

where

$$W(\hat{\omega})=u+\mathcal{U}^{(1)}(\hat{\omega}).$$

This solution justifies the *ansatz* (4.7), and we can identify

$$(m-\Delta|\alpha(\hat{\omega})|m)=\frac{(m-\Delta|W(\hat{\omega})|m)}{\mathcal{E}_m^{(0)}(\hat{\omega})-\mathcal{E}_{m-\Delta}^{(0)}(\hat{\omega})}, \quad (4.10a)$$

and

$$\beta_m(\omega)=-\tfrac{1}{2}\sum_\Delta \frac{(m-\Delta|W(\hat{\omega})|m)^2}{[\mathcal{E}_m^{(0)}(\hat{\omega})-\mathcal{E}_m^{(0)}(\hat{\omega})]^2}. \quad (4.10b)$$

The second-order eigenvalue is given by

$$\mathcal{E}_m^{(2)}(\hat{\omega})=(m|\mathcal{U}^{(2)}(\hat{\omega})|m)+\sum_\Delta \frac{(m-\Delta|W(\hat{\omega})|m)^2}{\mathcal{E}_m^{(0)}(\hat{\omega})-\mathcal{E}_{m-\Delta}^{(0)}(\hat{\omega})}. \quad (4.11)$$

By substituting (4.10a) into (4.8b) and relabeling some dummy indices, we obtain an integral equation for $(k|W(\hat{\omega})|k+\Delta)$:

$$\begin{aligned}(k|W(\hat{\omega})|k+\Delta)&=\tfrac{1}{2}+\int_m \hat{\rho}_m^{(0)}(km|T^{(1)}(\hat{\omega}+E_m^{(0)})|k+\Delta,m)\\&+\int_m\Big\{\hat{\rho}_m^{(0)}(km|T^{(0)}(\hat{\omega}+E_m^{(0)})|k+\Delta,m-\Delta)\\&\times\frac{(m-\Delta|W(E_m^{(0)})|m)}{\mathcal{E}_m^{(0)}(E_m^{(0)})-\mathcal{E}_{m-\Delta}^{(0)}(E_m^{(0)})}\\&+\hat{\rho}_{m-\Delta}^{(0)}(km|T^{(0)}(\hat{\omega}+E_{m-\Delta}^{(0)})|k+\Delta,m-\Delta)\\&\times\frac{(m-\Delta|W(E_{m-\Delta}^{(0)})|m)}{\mathcal{E}_{m-\Delta}^{(0)}(E_{m-\Delta}^{(0)})-\mathcal{E}_m^{(0)}(E_{m-\Delta}^{(0)})}\Big\}. \quad (4.12)\end{aligned}$$

This integral equation couples the matrix elements of W both in the momentum variables and in the two energy variables; all other quantities which appear are known from the solution of the uniform density problem. Note that W is real and symmetric in the momentum variables.

The needed matrix elements of $\mathcal{U}^{(2)}$ are obtained by combining (3.65) and (4.9)

$$\begin{aligned}(k|\mathcal{U}^{(2)}(\hat{\omega})|k)&=\int_m \hat{\rho}_m^{(0)}(km|T^{(2)}(\hat{\omega}+E_m^{(0)})|km)\\&+2\sum_\Delta\int_m \hat{\rho}_m^{(0)}\frac{(m-\Delta|W(E_m^{(0)})|m)}{E_m^{(0)}-\mathcal{E}_{m-\Delta}^{(0)}(E_m^{(0)})}\\&\times(km|T^{(1)}(\hat{\omega}+E_m^{(0)})|k,m-\Delta)\\&+\sum_\Delta\int_m \hat{\rho}_m^{(0)}\left[\frac{(m-\Delta|W(E_m^{(0)})|m)}{E_m^{(0)}-\mathcal{E}_{m-\Delta}^{(0)}(E_m^{(0)})}\right]^2\\&\times\{(k,m-\Delta|T^{(0)}(\hat{\omega}+E_m^{(0)})|k,m-\Delta)\\&-(km|T^{(0)}(\hat{\omega}+E_m^{(0)})|km)\}\\&+\int_m \hat{\rho}_m^{(2)}(km|T^{(0)}(\hat{\omega}+E_m^{(0)})|km)\\&+\int_m \hat{\rho}_m^{(0)}\left(km\left|\frac{\partial}{\partial\hat{\omega}}T^{(0)}(\hat{\omega}+E_m^{(0)})\right|km\right)E_m^{(2)}\\&-\int_m \rho_m^{(0)}(km|T^{(0)}(\hat{\omega}+E_m^{(0)})|km)\\&\times E_m^{(2)}\delta(\mu-E_m^{(0)}), \quad (4.13)\end{aligned}$$

where we have used $\mathcal{E}_m^{(0)}(E_m^{(0)})=E_m^{(0)}$. We have also introduced the notation

$$\hat{\rho}_m^{(2)}=\theta(\mu-E_m^{(0)})\rho_m^{(2)}.$$

We can now list the second-order, self-consistent equations as follows:

$$\mathcal{E}_m^{(2)}(\hat{\omega})=\sum_\Delta\frac{(m-\Delta|W(\hat{\omega})|m)^2}{\mathcal{E}_m^{(0)}(\hat{\omega})-\mathcal{E}_{m-\Delta}^{(0)}(\hat{\omega})}+(m|\mathcal{U}^{(2)}(\hat{\omega})|m), \quad (4.14a)$$

$$\rho_m^{(2)}=[\rho_m^{(0)}]^2\left[\frac{\partial\mathcal{E}_m^{(2)}}{\partial\hat{\omega}}+\frac{\partial^2\mathcal{E}_m^{(0)}}{\partial\hat{\omega}^2}E_m^{(2)}\right]_{\hat{\omega}=E_m^{(0)}}, \quad (4.14b)$$

$$E_m^{(2)}=\rho_m^{(0)}\mathcal{E}_m^{(2)}(E_m^{(0)}), \quad (4.14c)$$

where in obtaining (4.14c) we have used the relationship

$$E_m^{(2)}=\mathcal{E}_m^{(2)}(E_m^{(0)})+\frac{\partial\mathcal{E}_m^{(0)}(E_m^{(0)})}{\partial\hat{\omega}}E_m^{(2)}, \quad (4.15)$$

and that $\partial\mathcal{E}_m^{(0)}/\partial\hat{\omega}=1-\rho_m^{(0)-1}$.

Once the integral equation for the first-order quantities, (4.12), and the coupled integral equations for the second-order quantities, (4.13) and (4.14), have been solved, then the second-order energy and density, and the first-order density fluctuation can be evaluated from

$$\begin{aligned}\langle H\rangle^{(2)}&=\tfrac{1}{2}\int_k\Big\{\hat{\rho}_k^{(2)}[E_k^{(0)}+k^2]+\hat{\rho}_k^{(0)}E_k^{(2)}\\&-\rho_k^{(0)}E_k^{(2)}[E_k^{(0)}+k^2]\delta(\mu-E_k^{(0)})\\&-\sum_\Delta\hat{\rho}_k^{(0)}\frac{(k-\Delta|W(E_k)|k)}{E_k^{(0)}-\mathcal{E}_{k-\Delta}^{(0)}(E_k^{(0)})}\\&+\sum_\Delta\left[\frac{(k-\Delta|W(E_k^{(0)})|k)}{E_k^{(0)}-\mathcal{E}_{k-\Delta}^{(0)}(E_k^{(0)})}\right]^2\\&\times[(k-\Delta)^2-k^2]\Big\}, \quad (4.16a)\end{aligned}$$

$$\langle N\rangle^{(2)}=\int_k[\hat{\rho}_k^{(2)}-\rho_k^{(0)}E_k^{(2)}\delta(\mu-E_k^{(0)})], \quad (4.16b)$$

$$\langle Q\rangle^{(1)}=\sum_\Delta\int_k\hat{\rho}_k^{(0)}\frac{(k-\Delta|W(E_k^{(0)})|k)}{E_k^{(0)}-\mathcal{E}_{k-\Delta}^{(0)}(E_k^{(0)})}. \quad (4.16c)$$

Note that the constraining potential changes the density of the system to second order. This is because the chemical potential μ, which is also the separation energy of the last particle, has been fixed at its zero-order value. The change in energy per particle, at the zero-order density corresponding to μ, is then

$$\left(\frac{E}{N}\right)^{(2)}=\frac{\langle H\rangle^{(2)}}{\langle N\rangle^{(0)}}-\left(\frac{\partial\langle H\rangle^{(0)}}{\partial\langle N\rangle^{(0)}}\right)\frac{\langle N\rangle^{(2)}}{\langle N\rangle^{(0)}}. \tag{4.17}$$

V. CONCLUSIONS

The Martin-Schwinger-Puff equations for nuclear matter have been generalized to include the constraint of a spatial density ripple. With no new approximations beyond those already made for the uniform density case, a tractable set of coupled integral equations has been derived through second order in the density ripple. There appears to be no fundamental difficulty in extending the method to arbitrary finite order, although the complexities would undoubtedly increase rapidly.

The next objective of the current program is to perform numerical calculations in powers of the ripple wave number, q. The lowest order (q^0) gives an expression related to compressibility, and is not intrinsically interesting, since the same information is obtainable from the solutions of the uniform density problem; a solution to this order would, however, provide a numerical check on the procedures. The solution through order q^2 gives the results needed to evaluate the coefficient, $f(\rho)$, of the $|\nabla\rho|^2$ term in the energy expression (2.3). It is not possible to extend this simple procedure in order to evaluate the coefficients of the higher orders in the density gradient, because of the specific form of the constraint. (There are three independent fourth-order gradient terms. A calculation to order $a_0^2q^4$ gives only one more parameter.)

Once $f(\rho)$ has been evaluated, the methods of Berg and Wilets[10] could be used to compute properties of the nuclear surface, if the fourth-order terms are small. We have no plans to evaluate the fourth-order terms, but only judge the convergence of the expansion (for the nuclear surface) in terms of the magnitude of the second-order correction. If this appears to be very large, the usefulness of the expansion for the nuclear surface is obviated. If the correction is moderate, we could also investigate if it is sufficient to account for the discrepancy between experiment and the Brueckner, Lockett, and Rotenberg calculations on finite nuclei.

It should be borne in mind that really finite nuclei always possess distinctive properties, arising, for example, from their individual shell structures. Whereas the K matrix is intended to include configuration mixing (particularly from "distant" configurations), the kind of configuration mixing present in a small nucleus may be quite different in character from that assumed in calculating the K matrix. Thus doubly closed shell nuclei were chosen by Brueckner, Lockett, and Rotenberg because nearby configuration mixing is small, but the calculation of the reactance matrix did not assume anomalously small mixing of nearby configurations. In this respect, a deformed nucleus, where the shell-model degeneracies have been effectively broken, might yield better to this type of calculation. The questions raised in this paragraph are distinct from the problem to which we have addressed ourselves in this paper. (See, however, Eden, Emery, and Sampanthar.[4])

Finally we note that the many-body techniques used here do not include Bardeen-Cooper-Schrieffer[11] pairing correlations. One consequence of this omission, at least, is that the solutions might display Overhauser correlations if the internucleonic potential is sufficiently strong.[7] Nevertheless, it would be of interest to hunt for such solutions for $q\sim 2k_F$.

[10] R. A. Berg and L. Wilets, Phys. Rev. **101**, 1805 (1956); L. Wilets, Rev. Mod. Phys. **30**, 542 (1958).

[11] J. Bardeen, L. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957); N. N. Bogoliubov, V. V. Tolmochev, and D. V. Shirkov, *A New Method in the Theory of Superconductivity* (Consultants Bureau, New York, 1959).